Convergence - Solution of Equations (3 pages; 12/6/20)

(1) Recap of definitions:

1st order convergence: $e_{r+1} \approx ke_r$ (where $|k| < 1$)

2nd order convergence: $e_{r+1} \approx k{e_r}^2$

For the Fixed point method: $e_{r+1} \approx g'(\alpha)e_r$

So it has 1st order convergence, unless $g'(\alpha) = 0$.

(2) For the Fixed point method, $e_{r+1} \approx ke_r = g'(\alpha)e_r$

and $\frac{x_{r+1}-x_r}{x_r-x_{r-1}} \approx k$

Proof

$e_r = x_r - \alpha$ and $e_{r+1} = x_{r+1} - \alpha$

$e_r \approx ke_{r-1}$ and $e_{r+1} \approx ke_r$

So $\frac{x_{r+1}-x_r}{x_r-x_{r-1}} = \frac{(\alpha+e_{r+1})-(\alpha+e_r)}{(\alpha+e_r)-(\alpha+e_{r-1})} = \frac{e_{r+1}-e_r}{e_r-e_{r-1}} = \frac{ke_r-ke_{r-1}}{e_r-e_{r-1}} \approx k$

Alternatively, $g'(\alpha) \approx \frac{g(x_r)-g(x_{r-1})}{x_r-x_{r-1}} = \frac{x_{r+1}-x_r}{x_r-x_{r-1}}$

(3) Fixed point method when $g'(\alpha) = 0$

$g'(\alpha) \approx \frac{g(\alpha)-g(x_r)}{\alpha-x_r} \Rightarrow g'(\alpha)(\alpha - x_r) \approx g(\alpha) - g(x_r)$

$\Rightarrow g(x_r) \approx g(\alpha) - g'(\alpha)(\alpha - x_r)$

$= g(\alpha) + g'(\alpha)(x_r - \alpha)$

A better approximation can be shown to be

$$g(x_r) \approx g(\alpha) + g'(\alpha)(x_r - \alpha) + g''(\alpha)\frac{(x_r-\alpha)^2}{2!}$$

(from the Taylor expansion).

Then, if $g'(\alpha) = 0$, $g(x_r) \approx g(\alpha) + g''(\alpha)\frac{(x_r-\alpha)^2}{2!}$

and so $e_{r+1} = x_{r+1} - \alpha = g(x_r) - g(\alpha)$

$\approx g''(\alpha)\frac{(x_r-\alpha)^2}{2!} = \lambda(e_r)^2$

ie 2nd order convergence

(4) When there is 2nd order convergence, so that $e_{r+1} \approx \lambda(e_r)^2$, where $e_r = x_r - \alpha$, the ratio $\frac{x_{r+1}-x_r}{(x_r-x_{r-1})^2}$ can be shown to be approximately $-\lambda$

Proof

$$\frac{x_{r+1}-x_r}{(x_r-x_{r-1})^2} = \frac{(e_{r+1}+\alpha)-(e_r+\alpha)}{((e_r+\alpha)-(e_{r-1}+\alpha))^2}$$

$$= \frac{e_{r+1}-e_r}{(e_r-e_{r-1})^2}$$

$$\approx \frac{\lambda e_r^{\,2}-\lambda e_{r-1}^{\,2}}{(e_r-e_{r-1})^2}$$

$$= \frac{\lambda(e_r+e_{r-1})}{e_r-e_{r-1}}$$

$$= \frac{\lambda(e_r-e_{r-1})}{e_r-e_{r-1}} + \frac{2\lambda e_{r-1}}{e_r-e_{r-1}}$$

$$\approx \lambda + \frac{2\lambda e_{r-1}}{\lambda e_{r-1}^{\,2}-e_{r-1}}$$

$$= \lambda + \frac{2\lambda}{\lambda e_{r-1}-1}$$

$\rightarrow \lambda + \frac{2\lambda}{-1} = -\lambda$

(5) The Newton-Raphson is a special case of the fixed point method when $g'(\alpha) = 0$, and so has 2nd order convergence.

Proof

$x_{x+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Write $g(x) = x - \frac{f(x)}{f'(x)}$

Then $g'(x) = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{\left(f'(x)\right)^2}$

As $f(\alpha) = 0$, $g'(\alpha) = 1 - \frac{\left(f'(\alpha)\right)^2}{\left(f'(\alpha)\right)^2} = 0$