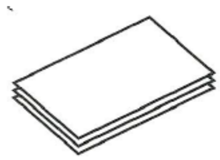
Configurations of 3 planes (11 pages; 1/8/21)

(1) Steps

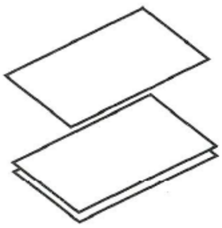
When investigating the configuration of 3 planes, the first step is to establish whether any of the planes are parallel.

These are the possible situations involving parallel planes:

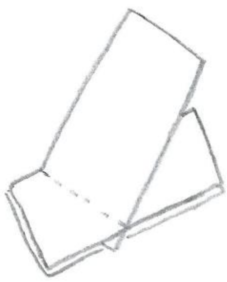
(A1) The 3 planes are identical (ie their equations are multiples of each other).



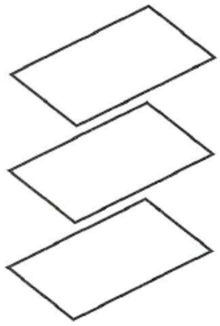
(A2) Two of the planes are identical, and the third is parallel to the other two (ie its direction vector is a multiple of that of each of the other two).



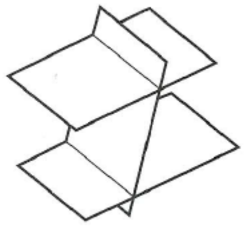
(A3) Two of the planes are identical, and the third is not parallel to the other two. The 3 planes meet in a straight line.



(A4) The 3 planes are parallel to each other.



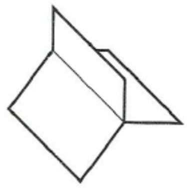
(A5) Two of the planes are parallel to each other, and the third is not parallel to the other two.



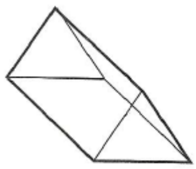
If none of the planes are parallel, then there are three possibilities:

(B) The planes intersect at a single point, and there is a unique solution to the associated simultaneous equations; ie the determinant for the left-hand side will be non-zero.

(C1) The planes meet in a straight line, forming a 'sheaf' of planes (as with the pages of a book, where the spine is the straight line). The associated simultaneous equations are 'consistent' (in general, this term covers all cases where there are one or more solutions).



(C2) The planes form a triangular prism (each pair of planes meet in a straight line, but there are no points common to all 3 planes - the associated simultaneous equations are 'inconsistent').



(2) Example

$x + 2y + 3z = 4 \quad (1)$

$2x - y + 4z = 1 \quad (2)$

$ax + 3y - z = b \quad (3)$

The 3 direction vectors are $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$ & $\begin{pmatrix} a \\ 3 \\ -1 \end{pmatrix}$, and we see that none of these vectors can be a multiple of either of the other two. So there are no parallel planes.

Then suppose that $\begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ a & 3 & -1 \end{vmatrix} = 0$,

so that $1(-11) - 2(-11) + a(11) = 0$ & hence $a = -1$.

Thus if $a \neq -1$, the planes will intersect at a single point.

If $a = -1$, then the planes will either form a sheaf or a triangular prism.

Approach 1

We can use one equation to eliminate one of the variables, and then ensure that the remaining equations are consistent (if the planes are to form a sheaf).

For example, (2) $\Rightarrow y = 2x + 4z - 1$

Then (1) & (3) give:

$x + 2(2x + 4z - 1) + 3z = 4$ & $-x + 3\,(2x + 4z - 1) - z = b$

so that $5x + 11z = 6$

and $5x + 11z = b + 3$

and hence $b = 3$

So when $b = 3$ (and $a = -1$), the planes form a sheaf, and when $b \neq 3$ (and $a = -1$), the planes form a triangular prism.

Approach 2

We can set x (for example) equal to a parameter λ, and use equations (1) & (2) to express y & z in terms of λ. Then we find the value of b such that equation (3) holds.

Thus (1) & (2) $\Rightarrow$

$2y + 3z = 4 - \lambda$ (4)

$-y + 4z = 1 - 2\lambda$ (5) $\Rightarrow -2y + 8z = 2 - 4\lambda$ (5′)

Adding (4) & (5):

$11z = 6 - 5\lambda \Rightarrow z = \frac{1}{11}(6 - 5\lambda)$

& (5) $\Rightarrow y = \frac{4}{11}(6 - 5\lambda) + 2\lambda - 1 = \frac{1}{11}(13 + 2\lambda)$

Then, substituting into (3):

$b = -\lambda + \frac{3}{11}(13 + 2\lambda) - \frac{1}{11}(6 - 5\lambda) = 3$

Approach 3

If there is to be a sheaf of planes, then we can first of all find the equation of the line of intersection of planes (1) & (2); L, say:

There will be a point on L with a z coordinate of 0 (choosing z because of the slightly larger coefficients, 3 & 4, which will now disappear).

Then (1) & (2) become:

$x + 2y = 4$

& $2x - y = 1$

[The planes might of course both be parallel to the z-axis (ie the

z-axis is parallel to a vector in the plane) and not have any z terms; in which case either x or y can be used.]

As a change from the usual methods, we could solve these by Cramer's rule:

$x = \frac{\begin{vmatrix} 4 & 2 \\ 1 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}} = \frac{-6}{-5} = \frac{6}{5}$ and $y = \frac{\begin{vmatrix} 1 & 4 \\ 2 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}} = \frac{-7}{-5} = \frac{7}{5}$

(which can be extended to larger numbers of equations)

Thus the point $\begin{pmatrix} 6/5 \\ 7/5 \\ 0 \end{pmatrix}$ lies on L.

The direction vector of L can be found by taking the vector product of the normals to the planes (1) & (2):

$\begin{vmatrix} \underline{i} & 1 & 2 \\ \underline{j} & 2 & -1 \\ \underline{k} & 3 & 4 \end{vmatrix} = \begin{pmatrix} 11 \\ 2 \\ -5 \end{pmatrix}$ [this is a vector perpendicular to the two normals, and therefore in the direction of each of the two planes, and so in the direction of their intersection, L]

Thus the equation of L is $\underline{r} = \begin{pmatrix} 6/5 \\ 7/5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 11 \\ 2 \\ -5 \end{pmatrix}$,

and we can find b by substituting for $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{6}{5} + 11\lambda \\ \frac{7}{5} + 2\lambda \\ -5\lambda \end{pmatrix}$

into the equation of the 3rd plane:

Thus $-\left(\frac{6}{5} + 11\lambda\right) + 3\left(\frac{7}{5} + 2\lambda\right) - (-5\lambda) = b$,

so that $3 + 0\lambda = b$;   (*)

ie if $b = 3$, then any point on L (ie any value of λ) will satisfy the original equations.

[If the planes form a triangular prism (if $b \neq 3$), then L will not intersect with the 3rd plane.

Note that the coefficient of λ in (*) is bound to equal 0, as it equals

$\begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix} . \begin{pmatrix} 11 \\ 2 \\ -5 \end{pmatrix}$, where $\begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix}$ is the normal to plane (3), and $\begin{pmatrix} 11 \\ 2 \\ -5 \end{pmatrix}$ is the direction of L. And these two vectors will be perpendicular to each other in both the sheaf of planes and triangular prism configurations.

Alternatively, $\begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ -1 & 3 & -1 \end{vmatrix} = 0$ (as there is not a unique sol'n to the original equations), and this is equivalent to

$\begin{pmatrix}-1\\3\\-1\end{pmatrix}.\left[\begin{pmatrix}2\\-1\\4\end{pmatrix}\times\begin{pmatrix}1\\2\\3\end{pmatrix}\right]=0$ [see the theory for the scalar triple product]

$\Leftrightarrow\begin{pmatrix}-1\\3\\-1\end{pmatrix}.\begin{pmatrix}-11\\-2\\5\end{pmatrix}=0\Leftrightarrow\begin{pmatrix}-1\\3\\-1\end{pmatrix}.\begin{pmatrix}11\\2\\-5\end{pmatrix}=0$]

Approach 4

If the equations had had a unique solution, we could have used Cramer's rule to find eg

$$x=\frac{\begin{vmatrix}4&2&3\\1&-1&4\\b&3&-1\end{vmatrix}}{\begin{vmatrix}1&2&3\\2&-1&4\\-1&3&-1\end{vmatrix}}$$

As the denominator is zero, the only way in which a solution could exist is if the numerator is also zero.

[$px=q$ only has a solution when $p=0$, if $q=0$ also]

Then $\begin{vmatrix}4&2&3\\1&-1&4\\b&3&-1\end{vmatrix}=0\Rightarrow 4(-11)-(-11)+b(11)=0$

$\Rightarrow b=3$

An alternative justification is as follows:

Consider the point on the intersection L of planes (2) & (3) where $z=0$

Then, from (1), (2) & (3):

$x+2y=4$

$2x - y = 1$

$-x + 3y = b \qquad (6)$

As the original equations have no unique solution, we know that

$\alpha(x + 2y) + \beta(2x - y) + \gamma(-x + 3y) = 0$

for some α, β, γ not all zero, and holding for all x, y & z

Then, equating coefficients of x, we have:

$\alpha(1) + \beta(2) + \gamma(-1) = 0 \quad (7)$

and equating coefficients of y:

$\alpha(2) + \beta(-1) + \gamma(3) = 0 \quad (7')$

Also, in order for there to be a solution when $z = 0$ (from (6)):

$\alpha(4) + \beta(1) + \gamma(b) = 0 \quad (7'')$

Then from (7), (7') & (7''), non-zero solutions for $\alpha, \beta, \gamma \Rightarrow$

$$\begin{vmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ 4 & 1 & b \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 1 & 2 & 4 \\ 2 & -1 & 1 \\ -1 & 3 & b \end{vmatrix} = 0 \,,$$

which is the "Cramer's rule" condition, applied to z

Notes

(i) Not all A Level examiners may recognise this method (although it has appeared in mark schemes from time to time), so it might be safest just to use it as a check (or mention "Cramer's rule").

(ii) In some cases, the determinant in the numerator could be zero, independently of b, so that it would be necessary to consider

$$\begin{vmatrix} 1 & 4 & 3 \\ 2 & 1 & 4 \\ -1 & b & -1 \end{vmatrix} \text{ or } \begin{vmatrix} 1 & 2 & 4 \\ 2 & -1 & 1 \\ -1 & 3 & b \end{vmatrix}$$

(so that a condition is placed on b)

(3) A different interpretation

Consider three planes A, B & C (with respective normals $\underline{a}$, $\underline{b}$ and $\underline{c}$), and suppose that A and B meet in a line L. Without loss of generality, $\underline{a}$ and $\underline{b}$ can be drawn on a sheet of paper on a horizontal surface, with L being vertical (see diagram).

(a) If the three planes meet in L (ie as a sheaf of planes), then C will lie in a vertical plane through P.

(b) If the three planes meet at the single point P, then C will be tilted relative to the paper.

(c) If the three planes do not all meet together, and C is parallel to (or identical to) either A or B, then C will lie in a vertical plane, not through P. [Situations (A3) and (A5).]

(d) If the three planes do not all meet together, and C is not parallel to either A or B, then C again lies in a vertical plane, not through P. [C has to be vertical, otherwise it would intersect with L at some point] This is the triangular prism.

When C is vertical, $\underline{c}$ can be drawn in the same plane as $\underline{a}$ and $\underline{b}$ (ie the plane of the paper).

Now consider the determinant formed from the normals to the three planes (in Approach 3 above this was $\begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ -1 & 3 & -1 \end{vmatrix}$). This will be zero when there isn't a single point of intersection of the planes.

This determinant can be shown to be equal to the scalar triple product $\underline{c}.(\underline{b}\times\underline{a})$

($\begin{pmatrix}-1\\3\\-1\end{pmatrix}.\left[\begin{pmatrix}2\\-1\\4\end{pmatrix}\times\begin{pmatrix}1\\2\\3\end{pmatrix}\right]$ in Approach 3).

This can be shown to be equal to the volume of the parallelepiped formed by the vectors $\underline{a}, \underline{b}$ & $\underline{c}$, which is zero if $\underline{a}, \underline{b}$ & $\underline{c}$ lie in the same plane; ie if C is vertical.

This occurs in the cases (a), (c) & (d), where there isn't a single point of intersection of the planes. And this is consistent with the determinant being zero when there isn't a single point of intersection of the planes.

(4) Another interpretation

Consider the following planes:

$3x - 2y - 4z = 1$

$2x - y = 2$

$x + y + 12z = 3$

None of the planes are parallel

Expanding by the 2nd row,

$$\begin{vmatrix}3 & -2 & -4\\2 & -1 & 0\\1 & 1 & 12\end{vmatrix} = -2(-20) + (-1)(40) = 0$$

Thus, the planes form either a sheaf or a triangular prism.

The 3 normals to the planes are $\begin{pmatrix} 3 \\ -2 \\ -4 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} 1 \\ 1 \\ 12 \end{pmatrix}$

If the planes form a triangular prism, a cross-sectional plane of the prism will be perpendicular to each of the 3 planes.

If the normal to this cross-sectional plane is $\begin{pmatrix} p \\ q \\ r \end{pmatrix}$,

then $\begin{pmatrix} 3 \\ -2 \\ -4 \end{pmatrix} . \begin{pmatrix} p \\ q \\ r \end{pmatrix} = 0$, and similarly for the other two planes.

Then

$3p - 2q - 4r = 0$

$2p - q = 0$

$p + q + 12r = 0$

In order for there to be a non-trivial sol'n (ie one other than

$p = q = r = 0$, it must be the case that $\begin{vmatrix} 3 & -2 & -4 \\ 2 & -1 & 0 \\ 1 & 1 & 12 \end{vmatrix} = 0$, as before.

Alternatively, if the planes form a sheaf (as for the pages of a book), then there will be a straight line (the spine of the book) that is perpendicular to each of the normals to the planes. So, once again $\begin{vmatrix} 3 & -2 & -4 \\ 2 & -1 & 0 \\ 1 & 1 & 12 \end{vmatrix} = 0$, and unfortunately this interpretation doesn't allow us to distinguish between the triangular prism and the sheaf.