Complex Numbers – Q9 – Practice/Y2/H (22/5/21)

Consider two roots of $z^n = cos\theta + isin\theta$:

$$z_r = \cos\left(\frac{\theta}{n} + \frac{2\pi r}{n}\right) + isin\left(\frac{\theta}{n} + \frac{2\pi r}{n}\right)$$

and $z_R = \cos\left(\frac{\theta}{n} + \frac{2\pi R}{n}\right) + isin\left(\frac{\theta}{n} + \frac{2\pi R}{n}\right)$

(i) Find the condition on n for z_R to equal -z_r for some R, and find R in terms of r.

(ii) Find the condition on θ for z_R to be the conjugate of z_r for some R, and find R in terms of r.

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Solution

(i) $\frac{\theta}{n} + \frac{2\pi R}{n} = \frac{\theta}{n} + \frac{2\pi r}{n} \pm \pi$

$\Rightarrow 2R = 2r \pm n$

$\Rightarrow R = r \pm \frac{n}{2}$

So n has to be even.

(ii) $\frac{\theta}{n} + \frac{2\pi R}{n} = -(\frac{\theta}{n} + \frac{2\pi r}{n})$

$\Rightarrow 2\pi R = -2\theta - 2\pi r$

$\Rightarrow R = -\frac{\theta}{\pi} - r$

So θ has to be either 0 or π (ie a $= cos\theta + isin\theta$ has to be real,

so that $z^n - a = 0$ has real coefficients)

If $\theta = 0$, then $R = -r$, and if $\theta = \pi$, then $R = -1 - r$

Example 1: $z^n = 1 = cos0, +isin0$

$$z_r = \cos\left(\frac{0}{n} + \frac{2\pi r}{n}\right) + isin\left(\frac{0}{n} + \frac{2\pi r}{n}\right)$$

And if $R = -r,\ z_R = \cos\left(\frac{0}{n} - \frac{2\pi r}{n}\right) + isin\left(\frac{0}{n} - \frac{2\pi r}{n}\right) = {z_r}^*$

Example 2: $z^n = -1 = cos\pi, +isin\pi$

$$z_r = \cos\left(\frac{\pi}{n} + \frac{2\pi r}{n}\right) + isin\left(\frac{\pi}{n} + \frac{2\pi r}{n}\right)$$

And if $R = -1 - r,\ z_R = \cos\left(\frac{\pi}{n} - \frac{2\pi}{n} - \frac{2\pi r}{n}\right) + isin\left(\frac{\pi}{n} - \frac{2\pi}{n} - \frac{2\pi r}{n}\right)$

$$= \cos\left(-\frac{\pi}{n} - \frac{2\pi r}{n}\right) + isin\left(-\frac{\pi}{n} - \frac{2\pi r}{n}\right) = {z_r}^*$$