Complex Numbers Q21 – Practice/M (13/12/22)

(i) Show geometrically that

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

When is there equality?

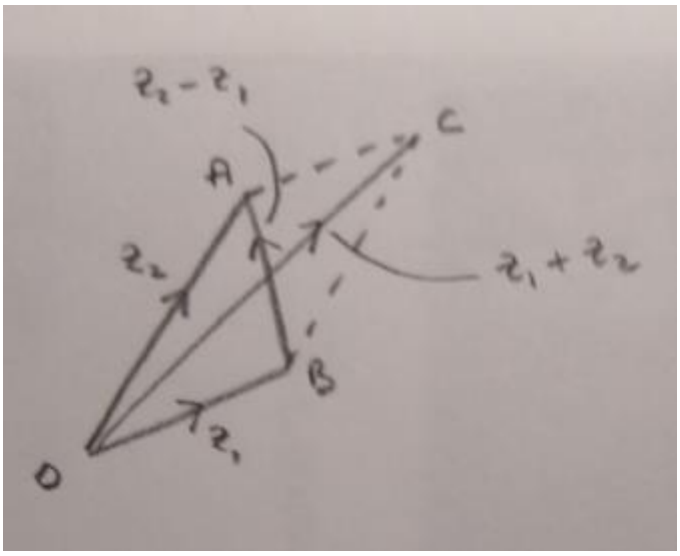
(ii) Show geometrically, and also from (i) that

$$|z_1 - z_2| \geq |z_1| - |z_2|$$

When is there equality?

Solution

(i)



Referring to the diagram, $|z_1 + z_2|$ is the length OC, whilst $|z_1|$ and $|z_2|$ are the lengths AC and OA. As $OC \leq OA + AC$,

the required result follows.

If $z_2 = kz_1$ (so that z_1 & z_2 have the same argument),

then $|z_1 + z_2| = |(1 + k)z_1| = (1 + k)|z_1|$

and $|z_1| + |z_2| = |z_1| + k|z_1| = (1 + k)|z_1|$

So there is equality when z_1 & z_2 have the same argument.

[Strictly speaking, we should also show that $|z_1 - z_2| = |z_1| + |z_2|$ means that $z_2 = kz_1$, and this can be seen geometrically, by requiring A to lie on OC.]

(ii) Referring to the diagram again, $|z_1 - z_2| = |z_2 - z_1|$ is the length BA.

Result to prove: $|z_1 - z_2| \geq |z_1| - |z_2|$; ie $BA \geq OB - OA$,

or $OB \leq OA + BA$, and this can be seen to be true from the diagram.

Alternatively, from (i): $|z_1 + z_2| \leq |z_1| + |z_2|$

or $|z_1| \geq |z_1 + z_2| - |z_2|$

So let $z_1 = u_1 - u_2$ and $z_2 = u_2$.

Then $|u_1 - u_2| \geq |(u_1 - u_2) + u_2| - |u_2|$

ie $|u_1 - u_2| \geq |u_1| - |u_2|$,

which can be rewritten as $|z_1 - z_2| \geq |z_1| - |z_2|$, as required.

Equality occurs when $|z_1 - z_2| = |z_1| - |z_2|$;

ie $|z_1| = |z_2| + |z_1 - z_2|$,

which is when $|z_1| \geq |z_2|$ and $z_1 = kz_2$, so that $k \geq 1$.