Complex Numbers – Q2 [6 marks](22/6/21)

Exam Boards

OCR : Pure Core (Year 1)

MEI: Core Pure (Year 1)

AQA: Pure (Year 1)

Edx: Core Pure (Year 1)

$1 + 3i$ is a root of the equation $z^3 + pz + q = 0$ (where p & q are real). Find the other roots, and the values of p & q [6 marks]

Solution

As the coefficients of the equation are real, the conjugate of

$1+3i$: $1-3i$ will also be a root. [1 mark]

Then the equation can be written as

$(z-[1+3i])(z-[1-3i])(z-\alpha)=0$, where α is the 3rd root.

[1 mark]

Expanding this gives $(z^2-2z+10)(z-\alpha)=0$ [1 mark]

and hence $z^3-(2+\alpha)z^2+(10+2\alpha)z-10\alpha=0$

Comparing the coefficients with those of $z^3+pz+q=0$,

we see that $\alpha=-2$, so that $p=6$ and $q=20$ [3 marks]

Alternative method

Using the standard results that the roots α, β & γ of the equation

$az^3+bz^2+cz+d=0$ satisfy $\alpha+\beta+\gamma=-\frac{b}{a}$, $\alpha\beta+\alpha\gamma+\beta\gamma=\frac{c}{a}$ and $\alpha\beta\gamma=-\frac{d}{a}$ (*):

$(1+3i)+(1-3i)+\alpha=0$ [since $b=0$]

Hence $\alpha=-2$

Also $(1+3i)(1-3i)-2(1+3i)-2(1-3i)=p$,

so that $10-2-2=p$ and $p=6$

And $-2(1+3i)(1-3i)=-q$,

so that $q=2(10)=20$

Notes

(a) A cubic function $y = f(x)$ **with real coefficients** will cross the x-axis at least once, and so $f(x) = 0$ has at least one real root (α, say). Then, factorising $f(x)$ as $(x - \alpha)g(x)$ means that, if β is a complex root of $f(x) = 0$, then β^*, the complex conjugate of β, must be the other root (considering the two roots derived from the quadratic formula).

[This could also have been written as $y = f(z)$ etc]

(b) (*) follows from expanding $(z - \alpha)(z - \beta)(z - \gamma) = 0$, and is in fact true whether the coefficients a, b & c are real or complex.