Circle Theorems (11 pages; 7/8/17)

Related theorems have been grouped together.

Theorem A1

The angle at the centre is twice the angle at the circumference (Figure 1)

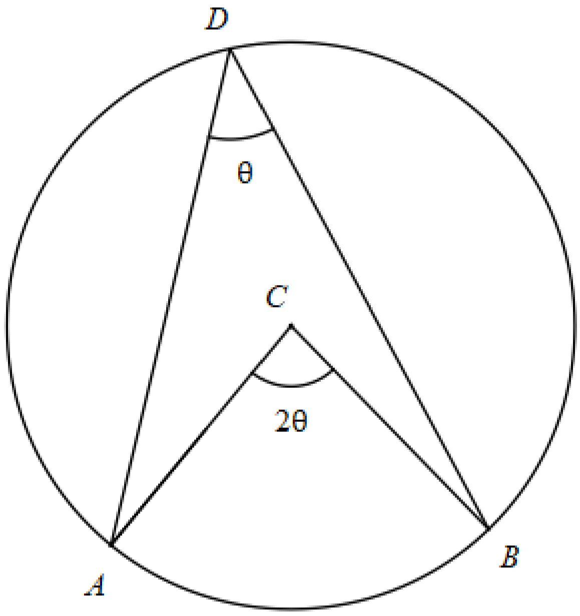

Figure 1

Exercise: By drawing in the radius CD, and using the isosceles triangles present, prove this theorem.

Solution (see Figure 2)

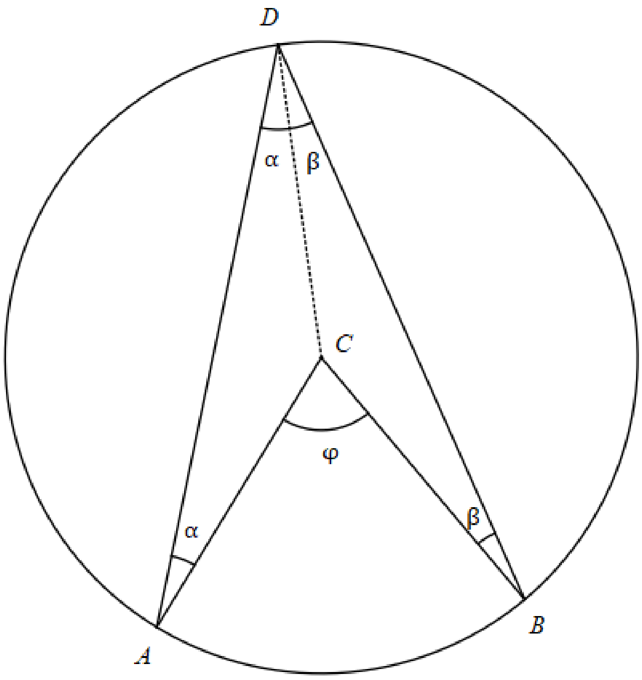

Figure 2

From the isosceles triangle ACD, we have the two angles α, and from the isosceles triangle BCD, we have the two angles β. We want to show that $\varphi = 2(\alpha + \beta)$.

Angle ACD is $180 - 2\alpha$, and angle BCD is $180 - 2\beta$, so that $(180 - 2\alpha) + (180 - 2\beta) + \varphi = 360$,

giving $-2\alpha - 2\beta + \varphi = 0$, and hence $\varphi = 2(\alpha + \beta)$, as required.

Theorem A2

The angle "in a semicircle" is always a right angle (Figure 3).

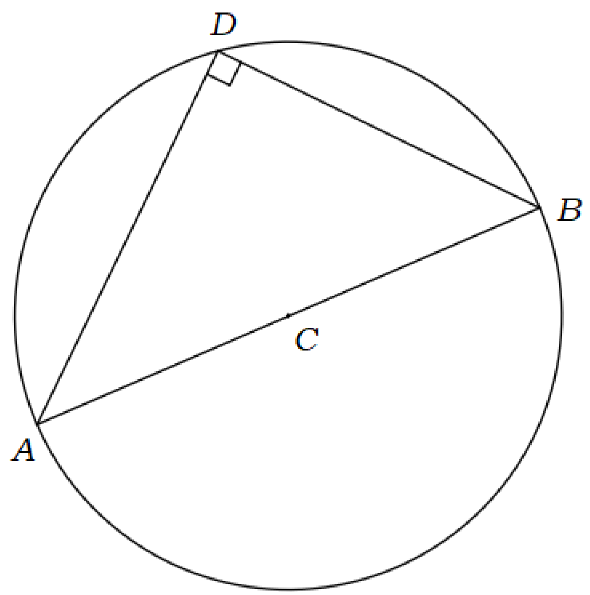

Figure 3

Figure 3 is a special case of Figure 2, where the angle ACB has been increased to 180°.

This theorem can also be proved directly by drawing in the radius CD, and considering the angles in the isosceles triangles.

Theorem A3

Angles "in the same segment" are equal (Figure 4).

[To explain the expression "in the same segment": the chord AB divides the circle into two segments.]

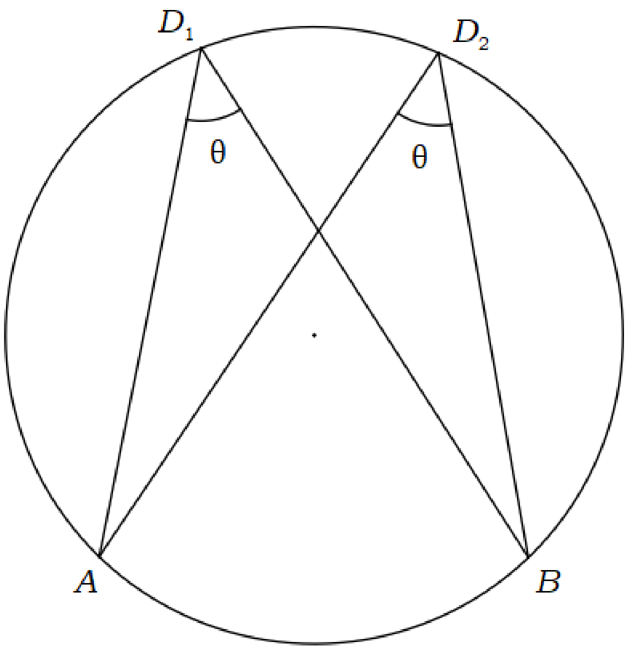

Figure 4

Theorem A3 follows from theorem A1, by considering two different points D, both of which will be half of the angle at the centre, and therefore equal to each other.

Result A4 (extension of Theorem 1)

Referring to Figure 1, if the angle ACB is increased so that it becomes greater than 180°, and the chord AB lies between C and D, then we have the situation shown below in Figure 5.

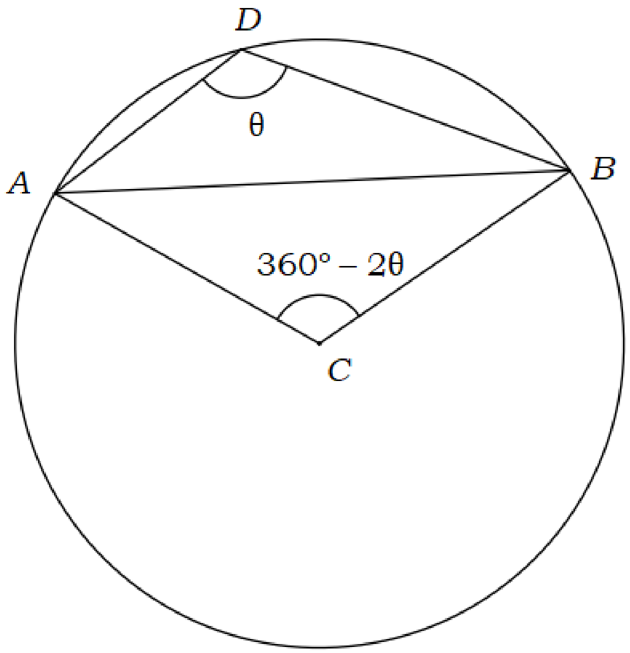

Figure 5

The larger angle ACB (greater than 180°) is still 2θ (strictly speaking, a new proof is required, as the situation has changed, but we will assume that the result of Theorem A1 still holds), so that the smaller angle ACB is $360° - 2\theta$.

Theorem A5

The opposite angles of a cyclic quadrilateral add up to 180° (Figure 6).

[A 'cyclic quadrilateral' is a quadrilateral, the corners of which lie on a circle.]

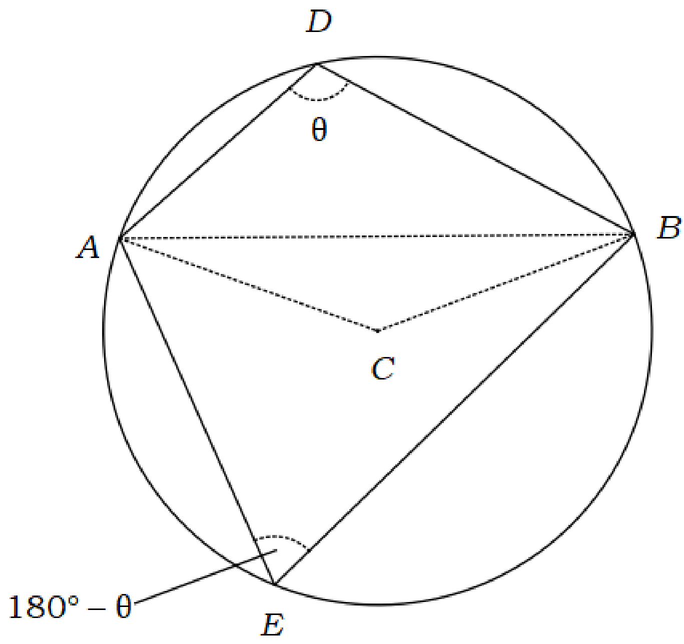

Figure 6

Proof: From Theorem A1, angle AEB will be a half of the smaller angle ACB, which was shown to be $360° - 2\theta$ in Result A4. Thus angle AEB is $180° - \theta$.

Theorem B6

The tangent to a circle at a point is perpendicular to the radius through that point (Figure 7).

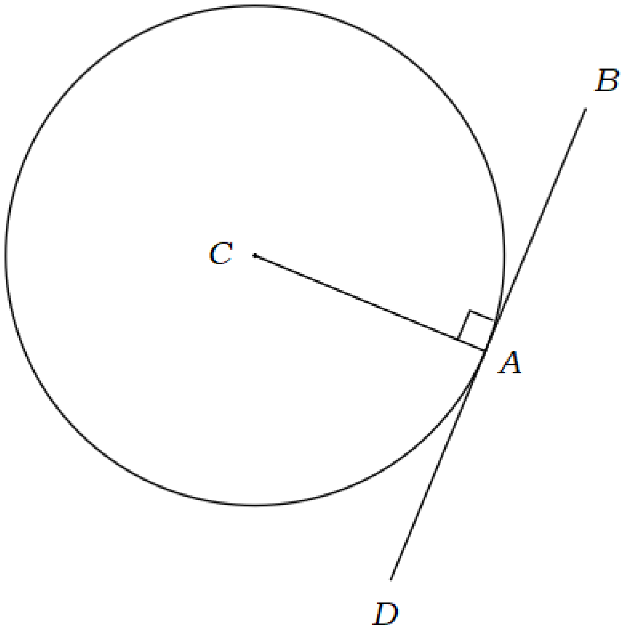

Figure 7

This result should be obvious from the symmetry of the situation: angle BAC must equal angle DAC.

Theorem B7 ("Alternate Segment Theorem")

Referring to Figure 8, the theorem says that angle FEA equals angle FAB (and also that angle EFA equals angle EAD). [The chord AF divides the circle into two 'alternate' segments, and similarly for AE.]

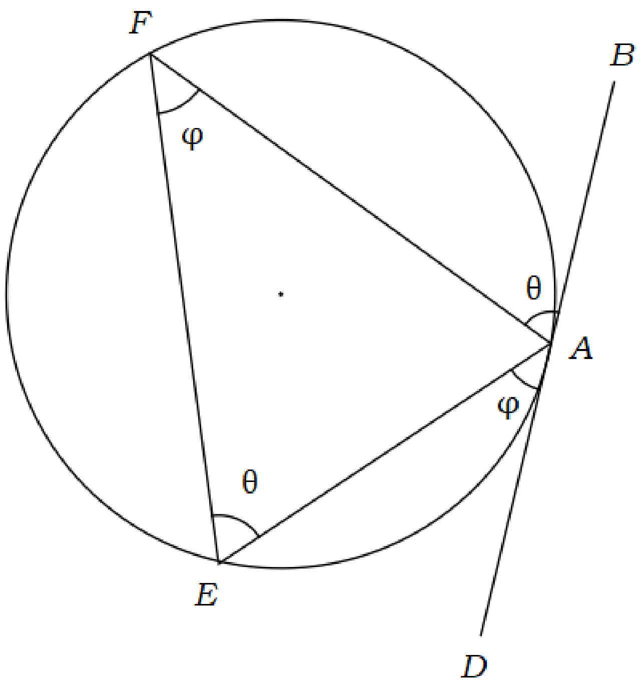

Figure 8

Exercise: By drawing in the radii CA, CE and CF, prove this theorem.

Solution (see Figure 9)

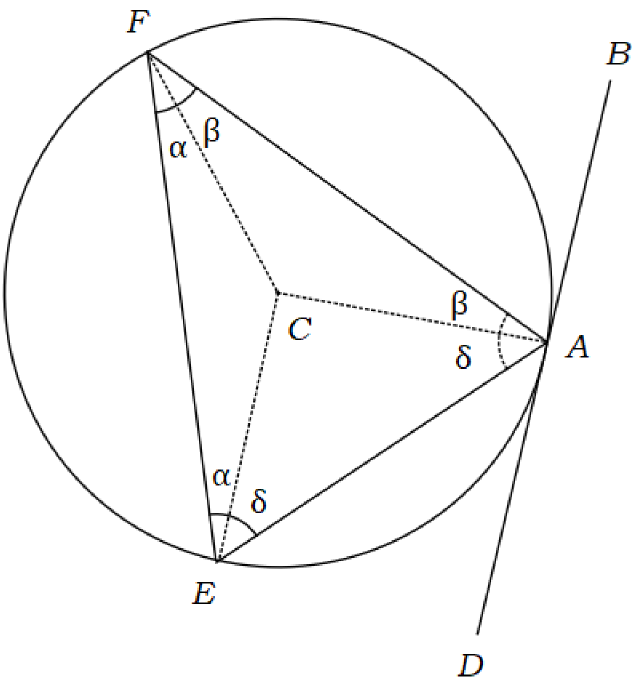

Figure 9

From the isosceles triangle CEF, we have the two angles α; from the isosceles triangle ACF, we have the two angles β, and from the isosceles triangle ACE, we have the two angles δ. We want to show that $\alpha + \delta = 90° - \beta$ (using theorem B6).

Adding together the angles in triangle AFE, we see that

$2\alpha + 2\beta + 2\delta = 180°$, so that $\alpha + \beta + \delta = 90°$ and hence $\alpha + \delta = 90° - \beta$.

Theorem B8

Referring to Figure 10, where AD and BD are tangents to the circle, the lengths AD and BD are equal.

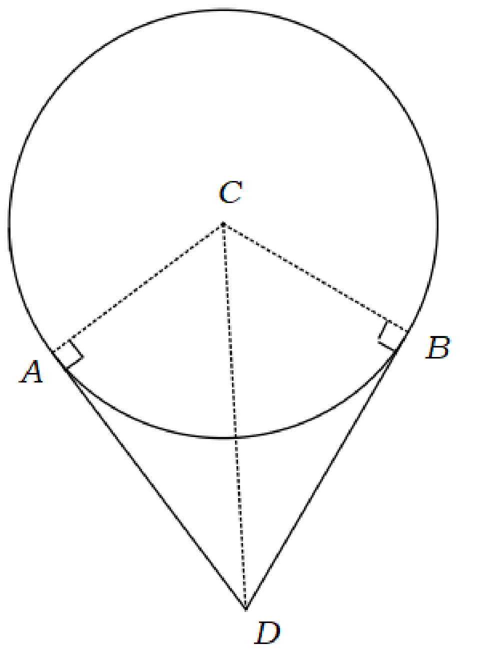

Figure 10

This follows from the fact that triangles CAD and CBD are congruent: Using theorem B6, they are both right-angled, share the hypotenuse CD, and sides AC and BC are equal.

Theorem C9

The perpendicular from the centre of a circle to a chord bisects the chord (Figure 11).

[The 'converse' is also true: the perpendicular bisector of a chord passes through the centre of the circle.]

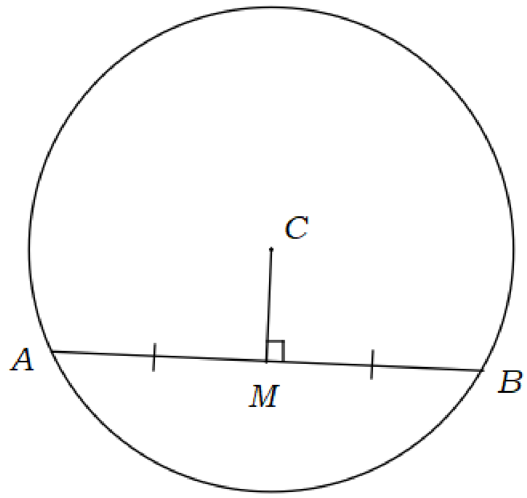

Figure 11

Exercise: By drawing in the radii CA, CE and CF, prove this theorem.

Solution (see Figure 12)

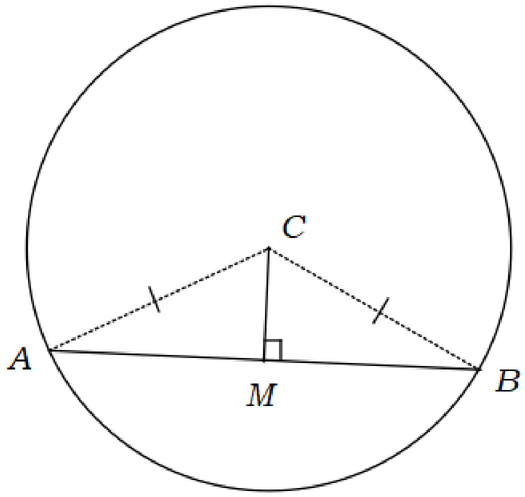

Figure 12

Triangles ACM and BCM are congruent, as they are both right-angled, share the side CM, and have the same hypotenuse. Therefore $AM = MB$, as required.