Arithmetic Sequences & Series (2 pages; 23/1/17)

(1) Terminology

$5, 7, 9, 11, 13 \ldots$ is an arithmetic sequence

$5 + 7 + 9 + 11 + 13 + \cdots + 43$ is an arithmetic series

(increasing by a 'common difference', 2)

(2) Ways of defining sequences eg $5, 7, 9, 11, 13 \ldots$

1st term, $a = 5$; common difference, $d = 2$

First Approach

kth term, $a_k = a + (k-1)d = 5 + 2(k-1)$

Second Approach

$a_{k+1} = a_k + 2$ (**inductive/iterative/recurrence** formula); $a_1 = 5$

(for $k \geq 2$)

Third Approach

$a_k = 2k + c$ (**deductive/direct** formula)

$a_1 = 5$, so $5 = 2(1) + c$ and hence $c = 3$ and $a_k = 2k + 3$

Compare with straight line $y = 2x + 3$

Common difference becomes the gradient

But constant term is $'a_0' = a_1 - 2 = 5 - 2 = 3$

(3) Sigma Notation

$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n$

$\sum_{k=1}^{n}(2k+3) = 5 + 7 + 9 + \cdots + (2n+3)$

(4) Proof of $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ (Gauss' method)

Let $\sum_{k=1}^{n} k = S$ Then $S = 1 + 2 + 3 + \cdots + n$

and also $S = n + (n-1) + (n-2) \ldots + 1$

As $1 + n = 2 + (n-1) = 3 + (n-2) = \cdots = n + 1$,

$2S = n(n+1)$ and so $\sum_{k=1}^{n} k = S = \frac{n(n+1)}{2}$

(5) Sum of general arithmetic series

Let $S = a + [a+d] + [a+2d] + \cdots + [a+(n-1)d]$

& $S = [a+(n-1)d] + [a+(n-2)d] + [a+(n-3)d] + \cdots + a$

So $2S = n[2a+(n-1)d]$ and $S = \frac{n}{2}[2a+(n-1)d]$

Alternative reasoning (informal):

As the terms go up at a steady rate, the sum will be the number of terms × the average term

average term is $\frac{1}{2}$(1st term + last term)

$= \frac{1}{2}(a + [a+(n-1)d])$

So $S = \frac{n}{2}[2a+(n-1)d]$