Algebra - Introduction (3 pages; 25/8/20)

(1) Many mathematical problems can be solved by expressing the given information in the form of one or more equations (or sometimes inequalities) - possibly using standard results such as Pythagoras' theorem - and then solving those equations. This is especially true of Mechanics.

(2) Avoiding division by zero [division by zero is 'undefined']

Example: Solve $tan\theta = sin\theta$, for $0 \leq \theta < 360$

Solution

$tan\theta = sin\theta \Rightarrow \frac{sin\theta}{cos\theta} = sin\theta$

Approach 1

Case 1: $sin\theta \neq 0$

$\Rightarrow \frac{1}{cos\theta} = 1$ [both sides can only be divided by $sin\theta$ if $sin\theta \neq 0$]

$\Rightarrow 1 = cos\theta$

$\Rightarrow \theta = 0$

Case 2: $sin\theta = 0$ (as this satisfies $\frac{sin\theta}{cos\theta} = sin\theta$)

$\Rightarrow \theta = 0 \; or \; 180$

Conclusion

$\theta = 0 \; or \; 180$

Approach 2

$tan\theta = sin\theta \Rightarrow \frac{sin\theta}{cos\theta} - sin\theta = 0$

$\Rightarrow sin\theta\left(\frac{1}{cos\theta} - 1\right) = 0$

$\Rightarrow sin\theta = 0\ \ (A)\ or\ \frac{1}{cos\theta} - 1 = 0\ (B)$

Then as for Approach 1.

(3) Rearranging an equation to get zero on one side

Example: Show that $\frac{1-tan^2\theta}{1+tan^2\theta} \equiv cos2\theta$

Solution

Result to prove: $\frac{1-tan^2\theta}{1+tan^2\theta} - cos2\theta \equiv 0$

LHS $= \frac{1-tan^2\theta}{sec^2\theta} - (cos^2\theta - sin^2\theta)$

$= cos^2\theta(1 - tan^2\theta) - cos^2\theta + sin^2\theta$

$= -cos^2\theta.\frac{sin^2\theta}{cos^2\theta} + sin^2\theta$

$= -sin^2\theta + sin^2\theta = 0$

[It is generally easier to work with a target of zero (ie rather than trying to rearrange the LHS into the RHS); especially if fractions are involved - as we need only show that the numerator is zero.]

(4) Correct use of $'\Rightarrow'$

Example: Show that $\frac{1-tan^2\theta}{1+tan^2\theta} = cos2\theta$ (A)

Avoid the following:

$\frac{1-tan^2\theta}{1+tan^2\theta} = cos2\theta$

$\Rightarrow \frac{1-tan^2\theta}{sec^2\theta} = cos^2\theta - sin^2\theta$

$\Rightarrow cos^2\theta(1 - tan^2\theta) = cos^2\theta - sin^2\theta$

$\Rightarrow -cos^2\theta tan^2\theta = -sin^2\theta$

$\Rightarrow -sin^2\theta \; = -sin^2\theta$

[This has only shown that the truth of $\frac{1-tan^2\theta}{1+tan^2\theta} = cos2\theta$ implies the truth of $-sin^2\theta \; = -sin^2\theta$, whereas we need to prove the opposite. The situation can sometimes be redeemed by the use of $' \Leftrightarrow '$ (if and only if) in place of $' \Rightarrow '$ (where this is valid of course), though the above layout is generally frowned on, as being inelegant. A compromise may be to show that the LHS and the RHS of (A) can both be shown to equal the same expression. Alternatively, each side could be rearranged, to produce a new equation, prior to another approach being adopted.]

(5) Factorise rather than expand

Example: To simplify $\frac{1}{2}n(n+1) + \frac{1}{6}n(n+1)(2n+1)$

Write $\frac{1}{6}n(n+1)\{3 + [2n+1]\} = \frac{1}{3}n(n+1)(n+2)$

(6) Identities

Note the distinction between $tan\theta = \frac{sin\theta}{cos\theta}$ (which is true for all values of θ) and $tan\theta = sin\theta$ (which is true only for certain values of θ). (Strictly speaking, we should write $tan\theta \equiv \frac{sin\theta}{cos\theta}$, to indicate that it is an 'identity'.)