# Second Order Linear Differential Equations with Constant Coefficients (9 pages; 28/5/20)

(1) Example 1: $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = x^2$   (*)

where $y = 1$  and $\frac{dy}{dx} = 2$ when $x = 0$

It has been found that a function of the form  $y = Ae^{\lambda x}$ will satisfy the **homogeneous** equation $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$.

[ $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = x^2$ is referred to as a **non-homogeneous** equation, and its solution will follow shortly.]

Substituting this function into (*) gives:

$\lambda^2 Ae^{\lambda x} - \lambda Ae^{\lambda x} - 6Ae^{\lambda x} = 0$,

which leads to the **auxiliary equation** $\lambda^2 - \lambda - 6 = 0$  or
$(\lambda + 2)(\lambda - 3) = 0$, which has roots $\lambda = -2$ & 3.

The general solution of  $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$ is then

$y = Ae^{-2x} + Be^{3x}$. (It can be verified that this satisfies (*), and it can be proved that it is the most general solution, by virtue of having two arbitrary constants.)

$y = Ae^{-2x} + Be^{3x}$ is referred to as the **complementary function** of  non-homogeneous equation.

(2) We now find a single solution of the non-homogeneous equation; say $y = f(x)$ (with no arbitrary constants).

It can then be seen that $y = Ae^{-2x} + Be^{3x} + f(x)$ satisfies the non-homogeneous equation, and once again the presence of two arbitrary constants means that it is the general solution.

(3) The single solution is referred to as a **particular integral**. (Not to be confused with 'particular solution', which refers to a solution of a differential equation that satisfies specified initial or boundary conditions). Its form will depend on the right-hand side of the non-homogeneous equation, and a table of **trial functions** is given in Appendix 1.

For the present example, where the right-hand side is $x^2$, the trial function is the quadratic function $y = ax^2 + bx + c$.

Substituting this into (*) gives

$2a - (2ax + b) - 6(ax^2 + bx + c) = x^2$

Equating the coefficients of powers of $x$:

$x^2: -6a = 1;\ \ x: -2a - 6b = 0\ ;\ \ \text{const. term: } 2a - b - 6c = 0$

leading to $a = -\frac{1}{6}, b = \frac{1}{18}\ , c = -\frac{7}{108}$

The general solution of the non-homogeneous equation is therefore

$y = Ae^{-2x} + Be^{3x} - \frac{x^2}{6} + \frac{x}{18} - \frac{7}{108}$

As $y = 1$ and $\frac{dy}{dx} = 2$ when $x = 0$,

$1 = A + B - \frac{7}{108}$

and $2 = -2A + 3B + \frac{1}{18}$

leading to $A = \frac{1}{4}$ and $B = \frac{22}{27}$

so that the particular solution is

$$y = \frac{1}{4}e^{-2x} + \frac{22}{27}e^{3x} - \frac{x^2}{6} + \frac{x}{18} - \frac{7}{108}$$

(4) Example 2: $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = cos2x$ (*)

The auxiliary equation is $\lambda^2 - 6\lambda + 9 = 0$ or $(\lambda - 3)^2 = 0$, which has a repeated root of $\lambda = 3$.

We need a general solution with two arbitrary constants (for a 2nd order differential equation), and it can be shown that a suitable complementary function in this case is $y = (A + Bx)e^{3x}$ (see Appendix 2).

The particular integral in this case has been found to be of the form $y = asin2x + bcos2x$.

Substituting into (*) gives

$$(-4asin2x - 4bcos2x) - 6(2acos2x - 2bsin2x)$$

$$+9(asin2x + bcos2x) = cos2x$$

Equating coefficients of $sin2x$ and $cos2x$:

$-4a + 12b + 9a = 0$ and $-4b - 12a + 9b = 1$

leading to $a = -\frac{12}{169}$ and $b = \frac{5}{169}$

The general solution is therefore

$$y = (A + Bx)e^{3x} - \frac{12sin2x}{169} + \frac{5cos2x}{169}$$

(5) Example 3: $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = e^{3x}$ (*)

The auxiliary equation is $\lambda^2 - 2\lambda + 5 = 0$, which has

roots $\lambda = 1 \pm 2i$

The complementary function is then $y = e^x(Ae^{2ix} + Be^{-2ix})$

$= e^x\{Acos2x + iAsin2x + Bcos(-2x) + iBsin(-2x)\}$

$= e^x\{(A + B)cos2x + i(A - B)sin2x\}$

$= e^x(Ccos2x + Dsin2x)$

[Note that $C$ and $D$ will be real if $B$ is the complex conjugate of $A$. As only real values of $C$ and $D$ are acceptable, it follows that $A$ and $B$ are restricted to be complex conjugates.]

The particular integral in this case has been found to be of the form $ae^{3x}$.

Substituting into (*) gives $9ae^{3x} - 2(3ae^{3x}) + 5ae^{3x} = e^{3x}$,

so that $8a = 1$ and $a = \frac{1}{8}$.

The general solution is therefore

$$y = e^x(Ccos2x + Dsin2x) + \frac{e^{3x}}{8}$$

**Note**: Functions such as $Ccos2x + Dsin2x$ can also be written in the form $Esin(2x + \alpha)$, for example.

(6) Example 4: $\frac{d^2y}{dx^2} + 9y = sin3x$ (*)

The auxiliary equation is $\lambda^2 + 9 = 0$, which has roots $\lambda = \pm 3i$

The complementary function is then $Asin3x + Bcos3x$.

A particular integral of the form $asin3x + bcos3x$ won't work (as the left-hand side will equal 0). We therefore adopt a trial function of $x(asin3x + bcos3x)$.

[For this reason, it is a good idea to find the complementary function before the particular integral.]

For this trial function,

$\frac{dy}{dx} = (asin3x + bcos3x) + x(3acos3x - 3bsin3x)$

and $\frac{d^2y}{dx^2} = 3acos3x - 3bsin3x + (3acos3x - 3bsin3x)$

$+x(-9asin3x - 9bcos3x)$

$= 6acos3x - 6bsin3x - 9x(asin3x + bcos3x)$

Then, substituting into (*);

$6acos3x - 6bsin3x - 9x(asin3x + bcos3x)$

$+9x(asin3x + bcos3x) = sin3x$

so that $6acos3x - 6bsin3x = sin3x$

Equating coefficients of $sin3x$ and $cos3x$:

$a = 0$ and $b = -\frac{1}{6}$

The general solution is therefore

$y = Asin3x + Bcos3x - \frac{1}{6}xcos3x$

(7) Particular Integral: Investigation

Explore the appropriate forms of the particular integral for the differential eq'n $\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = de^{kx}$, where $b, c, d \neq 0$, when the auxiliary eq'n has real roots.

**Solution**

Consider the trial function $y = (A + Bx + Cx^2)e^{kx}$

Then $\frac{dy}{dx} = (B + 2Cx + kA + kBx + kCx^2)e^{kx}$

and $\frac{d^2y}{dx^2} = (2C + kB + 2kCx + kB + 2kCx + k^2A + k^2Bx +$
$k^2Cx^2)e^{kx}$,

so that $\{2C + kB + 2kCx + kB + 2kCx + k^2A + k^2Bx + k^2Cx^2$

$+b[B + 2Cx + kA + kBx + kCx^2] + c[A + Bx + Cx^2]\}e^{kx}$

$= de^{kx}$

Equating coeffs of $x^2$: $k^2C + bkC + cC = 0$

$\Rightarrow C(k^2 + bk + c) = 0$ (1)

Equating coeffs of $x$: $2kC + 2kC + k^2B + 2bC + bkB + cB = 0$

ie $B(k^2 + bk + c) + 2C(2k + b) = 0$ (2)

Equating constant terms:

$2C + kB + kB + k^2A + b[B + kA] + cA = d$

$\Rightarrow A(k^2 + bk + c) + B(2k + b) + 2C = d$ (3)

**Case (i)**: $k$ isn't a root of the aux. eq'n (so that $k^2 + bk + c \neq 0$)

(1) $\Rightarrow C = 0$

Then (2) $\Rightarrow B = 0$

and (3) $\Rightarrow A = \frac{d}{A(k^2+bk+c)}$,

and so the particular integral is $y = \frac{d}{(k^2+bk+c)} e^{kx}$

**Case (ii)**: $k$ is a root of the aux. eq'n (so that $k^2 + bk + c = 0$)

and the aux. eq'n has repeated roots, so that $k = -\frac{b}{2}$

(3) $\Rightarrow C = \frac{d}{2}$

Also, the complementary function is of the form $y = (A + Bx)e^{kx}$, and this can be excluded from the particular integral (as it makes the LHS of the differential eq'n zero), whic is therefore

$y = \frac{d}{2} x^2 e^{kx}$

**Case (iii)**: $k$ is a root of the aux. eq'n (so that $k^2 + bk + c = 0$)

and the aux. eq'n doesn't have repeated roots, so that $k \neq -\frac{b}{2}$

(2) $\Rightarrow C = 0$

And (3) $\Rightarrow B(2k + b) = d$, so that $B = \frac{d}{2k+b}$

Also, the complementary function includes $y = Ae^{kx}$, and this can be excluded from the particular integral, which is therefore

$y = \frac{dx}{2k+b} e^{kx}$

## Summary

For $\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = de^{kx}$, where $b, c, d \neq 0$, when the auxiliary eq'n has real roots:

| | CF | PI |
|---|---|---|
| $k$ isn't a root of the aux. eq'n, which has distinct roots | $y = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$ | $y = \frac{d}{(k^2 + bk + c)}e^{kx}$ |
| $k$ isn't a root of the aux. eq'n, which has repeated roots | $y = (A + Bx)e^{\lambda x}$ | $y = \frac{d}{(k^2 + bk + c)}e^{kx}$ |
| $k$ is one of 2 distinct roots of the aux. eq'n | $y = Ae^{kx} + Be^{\lambda x}$ | $y = \frac{dx}{2k + b}e^{kx}$ |
| $k$ is a repeated root of the aux. eq'n | $y = (A + Bx)e^{kx}$ | $y = \frac{d}{2}x^2e^{kx}$ |

**Appendix 1**: Trial functions for the particular integral

| RHS of differential equation | Trial function |
|---|---|
| $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ ($a_n \neq 0$, but $a_i$ may be zero for $i \neq n$) | $b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0$ (if $a_i = 0$, we don't set $b_i = 0$) |
| $ae^{px}$ | $be^{px}$ |
| $a_1 sinpx + a_2 cospx$ (where one of the $a_i$ may be zero) | $b_1 sinpx + b_2 cospx$ (if $a_i = 0$, we don't set $b_i = 0$) |

**Notes**

(i) If the RHS is contained within the CF, include a factor of $x$ in the trial function.

(ii) If the CF is of the form $(A+Bx)e^{px}$, let the trial function be $Cx^2e^{px}$ (see Investigation above).

(iii) For a combination of functions on the RHS, the trial function will be the corresponding combination of trial functions; eg for

$x^2 + cos3x - 2e^{2x}$ it would be

$ax^2 + bx + c + csin3x + dcos3x + fe^{2x}$

(and equating coefficients of $x^2, x, x^0, sin3x, cos3x$ & $e^{2x}$)

**Appendix 2**: Complementary function where the auxiliary equation has repeated roots

Suppose that the homogeneous differential equation is

$\frac{d^2y}{dx^2} - 2k\frac{dy}{dx} + k^2y = 0$ , so that the auxiliary equation is

$\lambda^2 - 2k\lambda + k^2 = 0$ or $(\lambda - k)^2 = 0$

Then consider the complementary function $y = (A+Bx)e^{kx}$.

$\frac{dy}{dx} = Be^{kx} + k(A+Bx)e^{kx} = (B + kA + kBx)e^{kx}$

and $\frac{d^2y}{dx^2} = kBe^{kx} + k(B + kA + kBx)e^{kx}$ (differentiating the 1st expression for $\frac{dy}{dx}$)

$= (2kB + k^2A + k^2Bx)e^{kx}$

Then, substituting into the LHS of the differential equation gives

$(2kB + k^2A + k^2Bx)e^{kx} - 2k(B + kA + kBx)e^{kx}$

$+k^2(A+Bx)e^{kx}$

$= e^{\lambda x}\{Bx(k^2 - 2k^2 + k^2) + A(k^2 - 2k^2 + k^2) + B(2k - 2k)\}$

$= 0$ , as required.