Cambridge Assessment Admissions Testing

Sixth Term Examination Papers **9475**
MATHEMATICS 3 Morning
Wednesday 21 June 2023 Time: 3 hours

7041134096

Additional Material: Answer Booklet

INSTRUCTIONS TO CANDIDATES

Read this page carefully.

Do **NOT** open this question paper until you are told that you may do so.

Read and follow the additional instructions on the front of the answer booklet.

INFORMATION FOR CANDIDATES

There are 12 questions in this paper.

Each question is marked out of 20.

You may answer as many questions as you choose. You are advised to concentrate on no more than six questions. Little credit will be given for fragmentary answers.

All your answers will be marked.

Crossed out work will NOT be marked.

Your final mark will be based on the six questions for which you gain the highest marks.

There is NO Mathematical Formulae Booklet.

Calculators are NOT permitted.

Bilingual dictionaries are NOT permitted.

Wait to be told you may begin before turning this page.

Section A: Pure Mathematics

1 The distinct points $P(2ap,\, ap^2)$ and $Q(2aq,\, aq^2)$ lie on the curve $x^2 = 4ay$, where $a > 0$.

(i) Given that

$$(p+q)^2 = p^2q^2 + 6pq + 5\,, \qquad (*)$$

show that the line through P and Q is a tangent to the circle with centre $(0, 3a)$ and radius $2a$.

(ii) Show that, for any given value of p with $p^2 \neq 1$, there are two distinct real values of q that satisfy equation $(*)$.

Let these values be q_1 and q_2. Find expressions, in terms of p, for $q_1 + q_2$ and q_1q_2.

(iii) Show that, for any given value of p with $p^2 \neq 1$, there is a triangle with one vertex at P such that all three vertices lie on the curve $x^2 = 4ay$ and all three sides are tangents to the circle with centre $(0,\, 3a)$ and radius $2a$.

2 The polar curves C_1 and C_2 are defined for $0 \leqslant \theta \leqslant \pi$ by

$$r = k(1 + \sin\theta)$$

$$r = k + \cos\theta$$

respectively, where k is a constant greater than 1.

(i) Sketch the curves on the same diagram. Show that if $\theta = \alpha$ at the point where the curves intersect, $\tan\alpha = \dfrac{1}{k}$.

(ii) The region A is defined by the inequalities

$$0 \leqslant \theta \leqslant \alpha \quad \text{and} \quad r \leqslant k(1 + \sin\theta)\,.$$

Show that the area of A can be written as

$$\frac{k^2}{4}(3\alpha - \sin\alpha\cos\alpha) + k^2(1 - \cos\alpha)\,.$$

(iii) The region B is defined by the inequalities

$$\alpha \leqslant \theta \leqslant \pi \quad \text{and} \quad r \leqslant k + \cos\theta\,.$$

Find an expression in terms of k and α for the area of B.

(iv) The total area of regions A and B is denoted by R. The area of the region enclosed by C_1 and the lines $\theta = 0$ and $\theta = \pi$ is denoted by S. The area of the region enclosed by C_2 and the lines $\theta = 0$ and $\theta = \pi$ is denoted by T.

Show that as $k \to \infty$,

$$\frac{R}{T} \to 1$$

and find the limit of

$$\frac{R}{\mathrm{S}}$$

as $k \to \infty$.

3 **(i)** Show that, if a and b are complex numbers, with $b \neq 0$, and s is a positive real number, then the points in the Argand diagram representing the complex numbers $a+sb\mathrm{i}$, $a-sb\mathrm{i}$ and $a+b$ form an isosceles triangle.

Given three points which form an isosceles triangle in the Argand diagram, explain with the aid of a diagram how to determine the values of a, b and s so that the vertices of the triangle represent complex numbers $a+sb\mathrm{i}$, $a-sb\mathrm{i}$ and $a+b$.

(ii) Show that, if the roots of the equation $z^3+pz+q=0$, where p and q are complex numbers, are represented in the Argand diagram by the vertices of an isosceles triangle, then there is a non-zero real number s such that

$$\frac{p^3}{q^2} = \frac{27(3s^2-1)^3}{4(9s^2+1)^2}\,.$$

(iii) Sketch the graph $y = \dfrac{(3x-1)^3}{(9x+1)^2}$, identifying any stationary points.

(iv) Show that if the roots of the equation $z^3+pz+q=0$ are represented in the Argand diagram by the vertices of an isosceles triangle then $\dfrac{p^3}{q^2}$ is a real number and $\dfrac{p^3}{q^2} > -\frac{27}{4}$.

4 Let n be a positive integer. The polynomial p is defined by the identity

$$\mathrm{p}(\cos\theta) \equiv \cos\big((2n+1)\theta\big) + 1\,.$$

(i) Show that

$$\cos\big((2n+1)\theta\big) = \sum_{r=0}^{n} \binom{2n+1}{2r} \cos^{2n+1-2r}\theta(\cos^2\theta - 1)^r\,.$$

(ii) By considering the expansion of $(1+t)^{2n+1}$ for suitable values of t, show that the coefficient of x^{2n+1} in the polynomial $\mathrm{p}(x)$ is 2^{2n}.

(iii) Show that the coefficient of x^{2n-1} in the polynomial $\mathrm{p}(x)$ is $-(2n+1)2^{2n-2}$.

(iv) It is given that there exists a polynomial q such that

$$\mathrm{p}(x) = (x+1)\,[\mathrm{q}(x)]^2$$

and the coefficient of x^n in $\mathrm{q}(x)$ is greater than 0.

Write down the coefficient of x^n in the polynomial $\mathrm{q}(x)$ and, for $n \geqslant 2$, show that the coefficient of x^{n-2} in the polynomial $\mathrm{q}(x)$ is

$$2^{n-2}(1-n)\,.$$

5 **(i)** Show that if

$$\frac{1}{x} + \frac{2}{y} = \frac{2}{7}\,,$$

then $(2x-7)(y-7) = 49$.

By considering the factors of 49, find all the pairs of positive integers x and y such that

$$\frac{1}{x} + \frac{2}{y} = \frac{2}{7}\,.$$

(ii) Let p and q be prime numbers such that

$$p^2 + pq + q^2 = n^2$$

where n is a positive integer. Show that

$$(p+q+n)(p+q-n) = pq$$

and hence explain why $p + q = n + 1$.

Hence find the possible values of p and q.

(iii) Let p and q be positive and

$$p^3 + q^3 + 3pq^2 = n^3\,.$$

Show that $p + q - n < p$ and $p + q - n < q$.

Show that there are no prime numbers p and q such that $p^3 + q^3 + 3pq^2$ is the cube of an integer.

6 **(i)** By considering the Maclaurin series for e^x, show that for all real x,

$$\cosh^2 x \geqslant 1 + x^2.$$

Hence show that the function f, defined for all real x by $f(x) = \tan^{-1} x - \tanh x$, is an increasing function.

Sketch the graph $y = f(x)$.

(ii) Function g is defined for all real x by $g(x) = \tan^{-1} x - \frac{1}{2}\pi \tanh x$.

(a) Show that g has at least two stationary points.

(b) Show, by considering its derivative, that $(1+x^2)\sinh x - x\cosh x$ is non-negative for $x \geqslant 0$.

(c) Show that $\dfrac{\cosh^2 x}{1+x^2}$ is an increasing function for $x \geqslant 0$.

(d) Hence or otherwise show that g has exactly two stationary points.

(e) Sketch the graph $y = g(x)$.

7 **(i)** Let f be a continuous function defined for $0 \leqslant x \leqslant 1$. Show that

$$\int_0^1 \mathrm{f}(\sqrt{x})\,\mathrm{d}x = 2\int_0^1 x\mathrm{f}(x)\,\mathrm{d}x\,.$$

(ii) Let g be a continuous function defined for $0 \leqslant x \leqslant 1$ such that

$$\int_0^1 \big(\mathrm{g}(x)\big)^2\,\mathrm{d}x = \int_0^1 \mathrm{g}(\sqrt{x})\,\mathrm{d}x - \frac{1}{3}\,.$$

Show that $\displaystyle\int_0^1 \big(\mathrm{g}(x) - x\big)^2\,\mathrm{d}x = 0$ and explain why $\mathrm{g}(x) = x$ for $0 \leqslant x \leqslant 1$.

(iii) Let h be a continuous function defined for $0 \leqslant x \leqslant 1$ with derivative h' such that

$$\int_0^1 \big(\mathrm{h}'(x)\big)^2\,\mathrm{d}x = 2\mathrm{h}(1) - 2\int_0^1 \mathrm{h}(x)\,\mathrm{d}x - \frac{1}{3}\,.$$

Given that $\mathrm{h}(0) = 0$, find h.

(iv) Let k be a continuous function defined for $0 \leqslant x \leqslant 1$ and a be a real number, such that

$$\int_0^1 \mathrm{e}^{ax}\big(\mathrm{k}(x)\big)^2\,\mathrm{d}x = 2\int_0^1 \mathrm{k}(x)\,\mathrm{d}x + \frac{\mathrm{e}^{-a}}{a} - \frac{1}{a^2} - \frac{1}{4}\,.$$

Show that a must be equal to 2 and find k.

8 If

$$y = \begin{cases} \mathrm{k}_1(x) & x \leqslant b \\ \mathrm{k}_2(x) & x \geqslant b \end{cases}$$

with $\mathrm{k}_1(b) = \mathrm{k}_2(b)$, then y is said to be *continuously differentiable* at $x = b$ if $\mathrm{k}_1'(b) = \mathrm{k}_2'(b)$.

(i) Let $\mathrm{f}(x) = x\mathrm{e}^{-x}$. Verify that, for all real x, $y = \mathrm{f}(x)$ is a solution to the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0$$

and that $y = 0$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1$ when $x = 0$.

Show that $\mathrm{f}'(x) \geqslant 0$ for $x \leqslant 1$.

(ii) You are given the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2\left|\frac{\mathrm{d}y}{\mathrm{d}x}\right| + y = 0$$

where $y = 0$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1$ when $x = 0$. Let

$$y = \begin{cases} \mathrm{g}_1(x) & x \leqslant 1 \\ \mathrm{g}_2(x) & x \geqslant 1 \end{cases}$$

be a solution of the differential equation which is continuously differentiable at $x = 1$.

Write down an expression for $\mathrm{g}_1(x)$ and find an expression for $\mathrm{g}_2(x)$.

(iii) State the geometrical relationship between the curves $y = \mathrm{g}_1(x)$ and $y = \mathrm{g}_2(x)$.

(iv) Prove that if $y = \mathrm{k}(x)$ is a solution of the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + p\frac{\mathrm{d}y}{\mathrm{d}x} + qy = 0$$

in the interval $r \leqslant x \leqslant s$, where p and q are constants, then, in a suitable interval which you should state, $y = \mathrm{k}(c - x)$ satisfies the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - p\frac{\mathrm{d}y}{\mathrm{d}x} + qy = 0\,.$$

THIS QUESTION CONTINUES ON THE FACING PAGE.

(v) You are given the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\left|\frac{\mathrm{d}y}{\mathrm{d}x}\right| + 2y = 0$$

where $y = 0$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1$ when $x = 0$.

Let $\mathrm{h}(x) = \mathrm{e}^{-x}\sin x$. Show that $\mathrm{h}'\left(\frac{1}{4}\pi\right) = 0$.

It is given that $y = \mathrm{h}(x)$ satisfies the differential equation in the interval $-\frac{3}{4}\pi \leqslant x \leqslant \frac{1}{4}\pi$ and that $\mathrm{h}'(x) \geqslant 0$ in this interval.

In a solution to the differential equation which is continuously differentiable at $(n+\frac{1}{4})\pi$ for all $n \in \mathbb{Z}$, find y in terms of x in the intervals

(a) $\frac{1}{4}\pi \leqslant x \leqslant \frac{5}{4}\pi$,

(b) $\frac{5}{4}\pi \leqslant x \leqslant \frac{9}{4}\pi$.

Section B: Mechanics

9 Two particles, A of mass m and B of mass M, are fixed to the ends of a light inextensible string AB of length r and lie on a smooth horizontal plane. The origin of coordinates and the x- and y-axes are in the plane.

Initially, A is at (0, 0) and B is (r, 0). B is at rest and A is given an instantaneous velocity of magnitude u in the positive y direction.

At a time t after this, A has position (x, y) and B has position (X, Y). You may assume that, in the subsequent motion, the string remains taut.

(i) Explain by means of a diagram why

$$X = x + r\cos\theta$$
$$Y = y - r\sin\theta$$

where θ is the angle *clockwise* from the positive x-axis of the vector $\overrightarrow{AB}$.

(ii) Find expressions for $\dot{X}$, $\dot{Y}$, $\ddot{X}$ and $\ddot{Y}$ in terms of $\ddot{x}$, $\ddot{y}$, $\dot{x}$, $\dot{y}$, r, $\ddot{\theta}$, $\dot{\theta}$ and θ, as appropriate.

Assume that the tension T in the string is the only force acting on either particle.

(iii) Show that

$$\ddot{x}\sin\theta + \ddot{y}\cos\theta = 0$$
$$\ddot{X}\sin\theta + \ddot{Y}\cos\theta = 0$$

and hence that $\theta = \dfrac{ut}{r}$.

(iv) Show that

$$m\ddot{x} + M\ddot{X} = 0$$
$$m\ddot{y} + M\ddot{Y} = 0$$

and find $my + MY$ in terms of t and m, M, u, r as appropriate.

(v) Show that

$$y = \frac{1}{m+M}\left(mut + Mr\sin\left(\frac{ut}{r}\right)\right).$$

(vi) Show that, if $M > m$, then the y component of the velocity of particle A will be negative at some time in the subsequent motion.

10 A thin uniform beam AB has mass $3m$ and length $2h$. End A rests on rough horizontal ground and the beam makes an angle of 2β to the vertical, supported by a light inextensible string attached to end B. The coefficient of friction between the beam and the ground at A is μ.

The string passes over a small frictionless pulley fixed to a point C which is a distance $2h$ vertically above A. A particle of mass km, where $k < 3$, is attached to the other end of the string and hangs freely.

(i) Given that the system is in equilibrium, find an expression for k in terms of β and show that $k^2 \leqslant \dfrac{9\mu^2}{\mu^2+1}$.

(ii) A particle of mass m is now fixed to the beam at a distance xh from A, where $0 \leqslant x \leqslant 2$. Given that $k = 2$, and that the system is in equilibrium, show that

$$\frac{F^2}{N^2} = \frac{x^2+6x+5}{4(x+2)^2}\,,$$

where F is the frictional force and N is the normal reaction on the beam at A.

By considering $\frac{1}{3} - \dfrac{F^2}{N^2}$, or otherwise, find the minimum value of μ for which the beam can be in equilibrium whatever the value of x.

Section C: Probability and Statistics

11 Show that

$$\sum_{k=1}^{\infty} \frac{k+1}{k!} x^k = (x+1)\mathrm{e}^x - 1\,.$$

In the remainder of this question, n is a fixed positive integer.

(i) Random variable Y has a Poisson distribution with mean n. One observation of Y is taken. Random variable D is defined as follows. If the observed value of Y is zero then $D = 0$. If the observed value of Y is k, where $k \geqslant 1$, then a fair k-sided die (with sides numbered 1 to k) is rolled once and D is the number shown on the die.

(a) Write down $\mathrm{P}(D = 0)$.

(b) Show, from the definition of the expectation of a random variable, that

$$\mathrm{E}(D) = \sum_{d=1}^{\infty} \left[d \sum_{k=d}^{\infty} \left(\frac{1}{k} \cdot \frac{n^k}{k!} \mathrm{e}^{-n} \right) \right].$$

Show further that

$$\mathrm{E}(D) = \sum_{k=1}^{\infty} \left(\frac{1}{k} \cdot \frac{n^k}{k!} \mathrm{e}^{-n} \sum_{d=1}^{k} d \right).$$

(c) Show that $\mathrm{E}(D) = \frac{1}{2}(n + 1 - \mathrm{e}^{-n})$.

(ii) Random variables $X_1, X_2, \ldots, X_n$ all have Poisson distributions. For each $k \in \{1, 2, \ldots, n\}$, the mean of X_k is k.

A fair n-sided die, with sides numbered 1 to n, is rolled. When k is the number shown, one observation of X_k is recorded. Let Z be the number recorded.

(a) Find $\mathrm{P}(Z = 0)$.

(b) Show that $\mathrm{E}(Z) > \mathrm{E}(D)$.

12 A drawer contains n pairs of socks. The two socks in each pair are indistinguishable, but each pair of socks is a different colour from all the others. A set of $2k$ socks, where k is an integer with $2k \leqslant n$, is selected at random from this drawer: that is, every possible set of $2k$ socks is equally likely to be selected.

(i) Find the probability that, among the socks selected, there is no pair of socks.

(ii) Let $X_{n,k}$ be the random variable whose value is the number of pairs of socks found amongst those selected. Show that

$$\mathrm{P}(X_{n,k} = r) = \frac{\dbinom{n}{r}\dbinom{n-r}{2(k-r)}2^{2(k-r)}}{\dbinom{2n}{2k}}$$

for $0 \leqslant r \leqslant k$.

(iii) Show that

$$r\mathrm{P}(X_{n,k} = r) = \frac{k(2k-1)}{2n-1}\mathrm{P}(X_{n-1,k-1} = r-1)\,,$$

for $1 \leqslant r \leqslant k$, and hence find $\mathrm{E}(X_{n,k})$.